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October 1, 1946

Dr. Harry Polachek
Bureau of Ordnance Re2c
Navy Department
Rm. 0407
Washington, D. C.

Dear Dr. Polachek:

I am sorry to be so late in sending you this description of the thermo-couple problem.

We have tried all of the suggested finite difference formulas for simplified one-dimensional problems. (The results after 30 time steps were satisfactory.) We don't think that more than 8 decimal places need be carried in the work. At least 15,000 time steps are involved (if the intervals of time are chosen as in the accompanying note)- so that we might cut down the space to $0 \leq \xi \leq 4\ell$, $0 \leq \eta \leq 4\ell$ -in which case $L = 8\ell$ in (E) on pp. 7,8.

We feel that if Professor Aiken undertakes the job he will probably set it up in a more efficient way. If it is advisable that someone of our group visit Professor Aiken, we can make such an arrangement.

I hope that your heat-sensitivity problem has been satisfactorily solved. Thanking you for your patience, I remain

Yours sincerely,

Eugene Isaacson

EI:hs

Problem of Temperature Measurement

We are interested in determining how the flow of heat in a solid conductor is perturbed by the insertion of a thermo-couple.

For instance, we consider a special problem connected with rockets. A cylindrical thermo-couple (T.C.) is fitted snugly into a cylindrical hole drilled in the wall of a rocket. The propellant is ignited and the hot gases heat up the wall of the rocket.

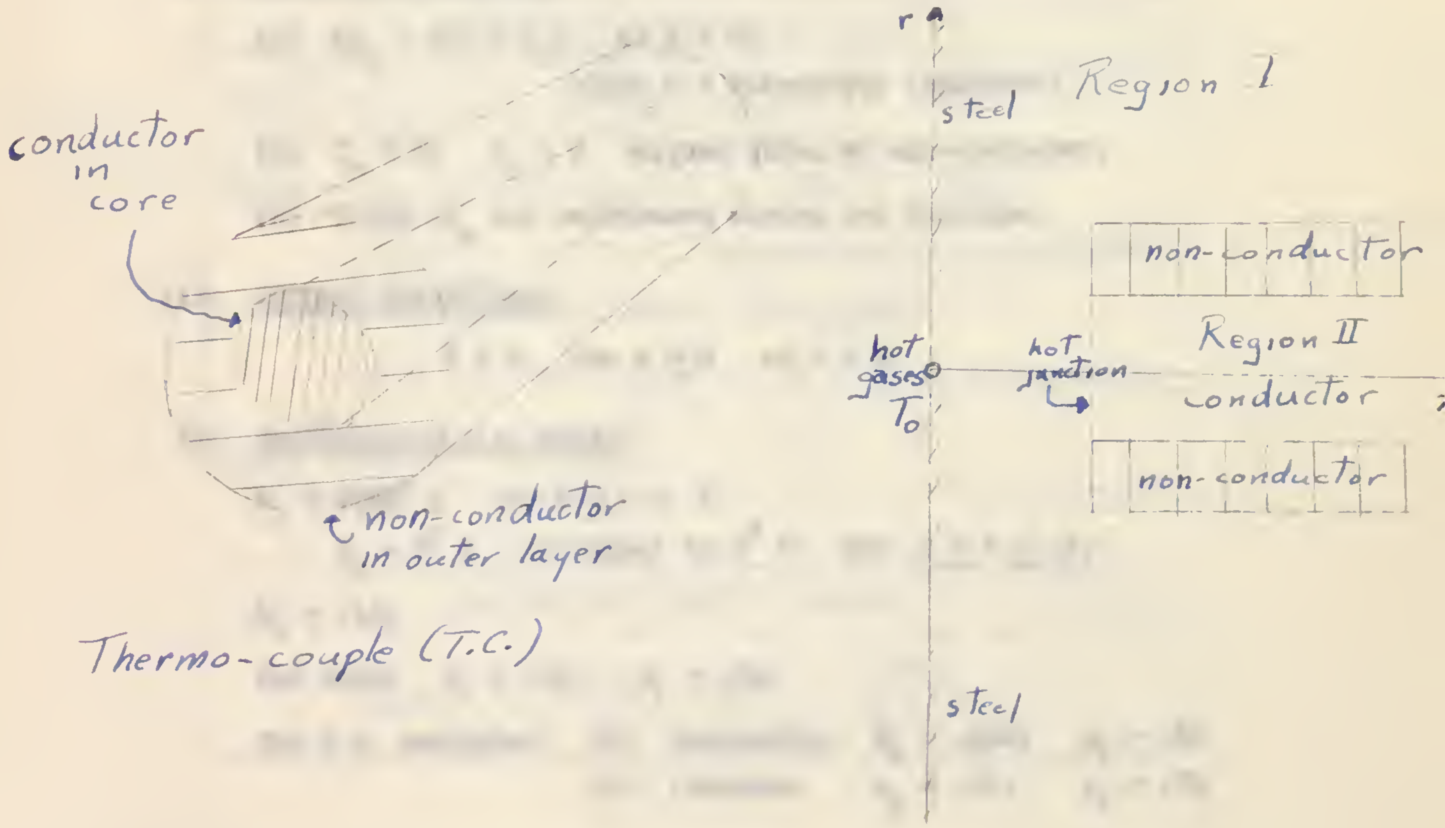

Fig. 1

The mathematical formulas describing the flow of heat through the steel wall and in and around the thermo-couple are

I. Partial D.E.:

$$k[T_{xx} + T_{rr} + \frac{1}{r} T_r] = cT_t ,$$

where T = Temperature $^\circ$K,
t = time,
x = distance along axis of T.C.,
r = radial distance from axis of T.C.,
k = heat conductivity (constant),
c = heat capacity (constant).

II. Boundary Conditions:

(a) $kT_x = h(T - T_o)$ at $x = 0$,
where h = Emissivity (constant),

(b) $T_x = 0$; $T_r = 0$ across sides of non-conductor,

(c) T and kT_x are continuous across hot junction.

III. Initial Conditions:

$$T = 0 \quad \text{for } x \geq 0 \quad \text{at } t = 0.$$

IV. Constants--c.g.s. units:

$T_o = 3000^\circ$ K for $0 \leq t < .1$,
$T_o = 0^\circ$ K (or linear to 0° K) for $.1 \leq t \leq .2$;

$h_1 = .18$;

for steel $k_1 = .11$, $c_1 = .84$;

for T.C. conductor (a) Constantin: $k_2 = .056$, $c_2 = .84$
(b) Platimum: $k_3 = .17$, $c_3 = .71$

Suggestions for Numerical Solution of Problem

(A) **Introduce dimensionless quantities:**

$$u = \frac{T}{3000};$$

$$\xi = \frac{h_1}{k_1} x, \quad \eta = \frac{h_1}{k_1} r \quad (h_1,\ k_1,\ c_1 \text{ of region I});$$

$$\tau = \frac{h_1^2}{k_1 c_1} t.$$

Equations in Region I become

P.D.E.: $u_{\xi\xi} + u_{\eta\eta} + \frac{1}{\eta} u_{\eta} = u_{\tau}$

B.C.: $u_{\xi} = u - 1$ at $\xi = 0$ (for $\tau \leq .035$)

$u_{\xi} = 0$, $u_{\eta} = 0$ across sides of non-conductor

u is continuous across hot-junction

$h_1 u_{\xi}$ on left $= k_2 \frac{h_1}{k_1} u_{\xi}$ on right of hot-junction

I.C.: $u = 0$ for $\xi \geq 0$ at $\tau = 0$

Equation in Region II becomes

P.D.E.: $u_{\xi\xi} + u_{\eta\eta} + \frac{1}{\eta} u_{\eta} = \frac{c_2}{k_2} \frac{k_1}{c_1} u_{\tau}$

(B) Cut off size of domain and use symmetry about ξ-axis.

In order to work with a finite set of net-points, we introduce the two additional boundaries $\xi = 5\ell$ and $\eta = 5\ell$, together with the corresponding boundary conditions $u_\eta = 0$ and $u_\xi = 0$. Therefore, the region of interest for computation is defined by

$$0 \leq \xi \leq 5\ell \quad \text{and} \quad 0 \leq \eta \leq 5\ell, \quad \ell = .012.$$

See Figure 2 on page 9.

(C) Size of space mesh:

$$\Delta\xi = \Delta\eta = \frac{\ell}{4} = .003,$$

$$\ell = .012.$$

(D) Finite difference equations:

(1) At non-boundary points in Region I

$u^\tau_{\xi,\eta}$ represents values of u at ξ, η, τ.

$$\frac{u^0_{-1,0} - 2u^0_{0,0} + u^0_{1,0}}{(\Delta\xi)^2} + \frac{u^0_{0,-1} - 2u^0_{0,0} + u^0_{0,1}}{(\Delta\eta)^2}$$

$$+ \frac{1}{\eta_0} \frac{[u^0_{0,+1} - u^0_{0,-1}]}{2\Delta\eta} = \frac{u^1_{0,0} - u^0_{0,0}}{\Delta\tau},$$

$\therefore$

$$u^1_{0,0} = u^0_{0,0} + \frac{\Delta\tau}{(\Delta\xi)^2}[u^0_{-1,0} + u^0_{1,0} + u^0_{0,-1} + u^0_{0,1} - 4u^0_{0,0}]$$

$$+ \frac{\Delta\eta\,\Delta\tau}{2\eta_0(\Delta\eta)^2}[u^0_{0,1} - u^0_{0,-1}].$$

Let $\dfrac{\Delta\tau}{(\Delta\xi)^2} = \dfrac{\Delta\tau}{(\Delta\eta)^2} = \dfrac{1}{4}$, then

$$u^1_{0,0} = \frac{1}{4}[u^0_{-1,0} + u^0_{1,0} + u^0_{0,-1} + u^0_{0,1}] + \frac{\Delta\eta}{8\,\eta_0}[u^0_{0,1} - u^0_{0,-1}]$$

or

$$u^1_{0,0} = \frac{1}{4}[u^0_{-1,0} + u^0_{1,0} + u^0_{0,-1} + u^0_{0,1}] + \frac{.003}{8\,\eta_0}[u^0_{0,1} - u^0_{0,-1}].$$

(2) At non-boundary points in Region II

Let $\frac{c_2}{k_2}\ \frac{k_1}{c_1} = \frac{1}{\alpha}$, then

$$u^1_{0,0} = u^0_{0,0} + \frac{\alpha}{4}[u^0_{-1,0} + u^0_{1,0} + u^0_{0,-1} + u^0_{0,1} - 4u^0_{0,0}] + \frac{\alpha\Delta\eta}{8\,\eta_0}[u^0_{0,1} - u^0_{0,-1}].$$

(3) At boundary points on line $\eta = 0$

Use $u^0_{0,-1} = u^0_{0,1}$

and the P.D.E.

Region I: $u_{\xi\xi} + 2u_{\eta\eta} = u_{\tau}$,

Region II: $u_{\xi\xi} + 2u_{\eta\eta} = \frac{c_2}{k_2}\ \frac{k_1}{c_1}\, u_{\tau}$.

$\therefore$ For Region I:

$$u^1_{0,0} = u^0_{0,1} + \frac{1}{4}[u^0_{-1,0} - 2u^0_{0,0} + u^0_{1,0}];$$

for Region II:

$$u^1_{0,0} = \alpha u^0_{0,1} + (1 - \alpha)u^0_{0,0} + \frac{\alpha}{4}[u^0_{-1,0} - 2u^0_{0,0} + u^0_{1,0}].$$

(4) At boundary points on line $\xi = 0$
(Except for points at corners)

We use the boundary condition $u_\xi = u - 1$. Now,

$$u^0_{0,0} + \Delta\xi\left[\frac{u^0_{0,0} - 1}{2} + \frac{u^0_{2,0} - u^0_{0,0}}{4\,\Delta\xi}\right] = u^0_{1,0}.$$

(Since [ ] = u_ξ at $u^0_{\frac{1}{2},0}$.)

Therefore,

$$u^0_{0,0} = \frac{u^0_{1,0} - \frac{1}{4}u^0_{2,0} + \frac{\Delta\xi}{2}}{\frac{3}{4} + \frac{\Delta\xi}{2}}.$$

(5) At boundary points where either $u_\xi = 0$ or $u_\eta = 0$
(Except for corner points)

Use the same scheme as in (4) replacing the derivative at the boundary point by 0. For example, on the line $\eta = \ell$

$$u^0_{0,0} + \Delta\eta\left[0 + \frac{u^0_{0,2} - u^0_{0,0}}{4\,\Delta\eta}\right] = u^0_{0,1}.$$

So that,

$$u^0_{0,0} = \frac{4}{3}\left[u^0_{0,1} - \frac{u^0_{0,2}}{4}\right].$$

(6) At the junction $\xi = \ell$

Use the boundary condition: u is continuous, and $h_1 u_\xi$ on left $= \frac{k_2}{k_1} h_1 u_\xi$ on right.

$$u_\xi \text{ on left} = -\frac{1}{2\,\Delta\xi}[4u^0_{-1,0} - 3u^0_{0,0} - u^0_{-2,0}],$$

$$u_\xi \text{ on right} = \frac{1}{2\,\Delta\xi}[4u^0_{1,0} - 3u^0_{0,0} - u^0_{2,0}],$$

whence

$$u^{o}_{0,0} = \frac{\frac{k_2}{k_1}[4u^{o}_{1,0} - u^{o}_{2,0}] + [4u^{o}_{-1,0} - u^{o}_{-2,0}]}{3\left(1 + \frac{k_2}{k_1}\right)} .$$

(7) At corner boundary points

Use average of the two values which are found by the two corresponding methods previously described.

(8) Computation of solution when thermo-couple is absent:

Since we have cut off the domain in (5), we find that the solution of the problem

P.D.E.: $U_{\xi\xi} = U_{\tau}$ for $0 \leq \xi \leq 5\ell$;

B.C.: $U_{\xi} = U - 1$ at $\xi = 0$,

$U_{\xi} = 0$ at $\xi = 5\ell$;

I.C.: $U = 0$ at $\tau = 0$,

will yield the description of the flow of heat when no thermo-couple is present. The solution of this simple problem can be found in "Conduction of Heat in Solids," by H. S. Carslaw, 2nd edition, pp. 74-79. We merely write down the formulas:

$$U(\xi, \tau) = 1 + w(\xi, \tau)$$

$$\text{where} \quad w(\xi, \tau) = \sum_{1}^{\infty} A_n X_n e^{-a_n^2 \tau}$$

with $a_n = n^{\text{th}}$ positive root of $\left[\tan aL = \frac{2a}{a^2 - 1}\right]$,

$$X_n(\xi) = \cos a_n \xi + \frac{1}{a_n} \sin a_n \xi ,$$

$$A_n = \frac{-2a_n^2}{(a_n^2 + 1)L + 2} \int_0^L X_n(\xi)d\xi ,$$

$$L = 10\ell .$$

A check on some of these calculations is

$$-1 = \sum_1^\infty A_n X_n(\xi).$$

$u(\xi, \eta, \tau)$ should be compared with $U(\xi, \tau)$ at all of the ξ, η net-points near the hot junction (say $0 \leq \xi \leq 2\ell$, $0 \leq \eta \leq 2\ell$).

(F) Order of problems:

Realizing that this job is not trivial, we would be satisfied if the case of Constantin could be carried out up to $\tau = .035$ approximately. If it is feasible to continue beyond this point, then the boundary condition $u_\xi = u$ should be introduced at $\xi = 0$ for $.035 < \tau < .070$ approximately.

Schematic Diagram

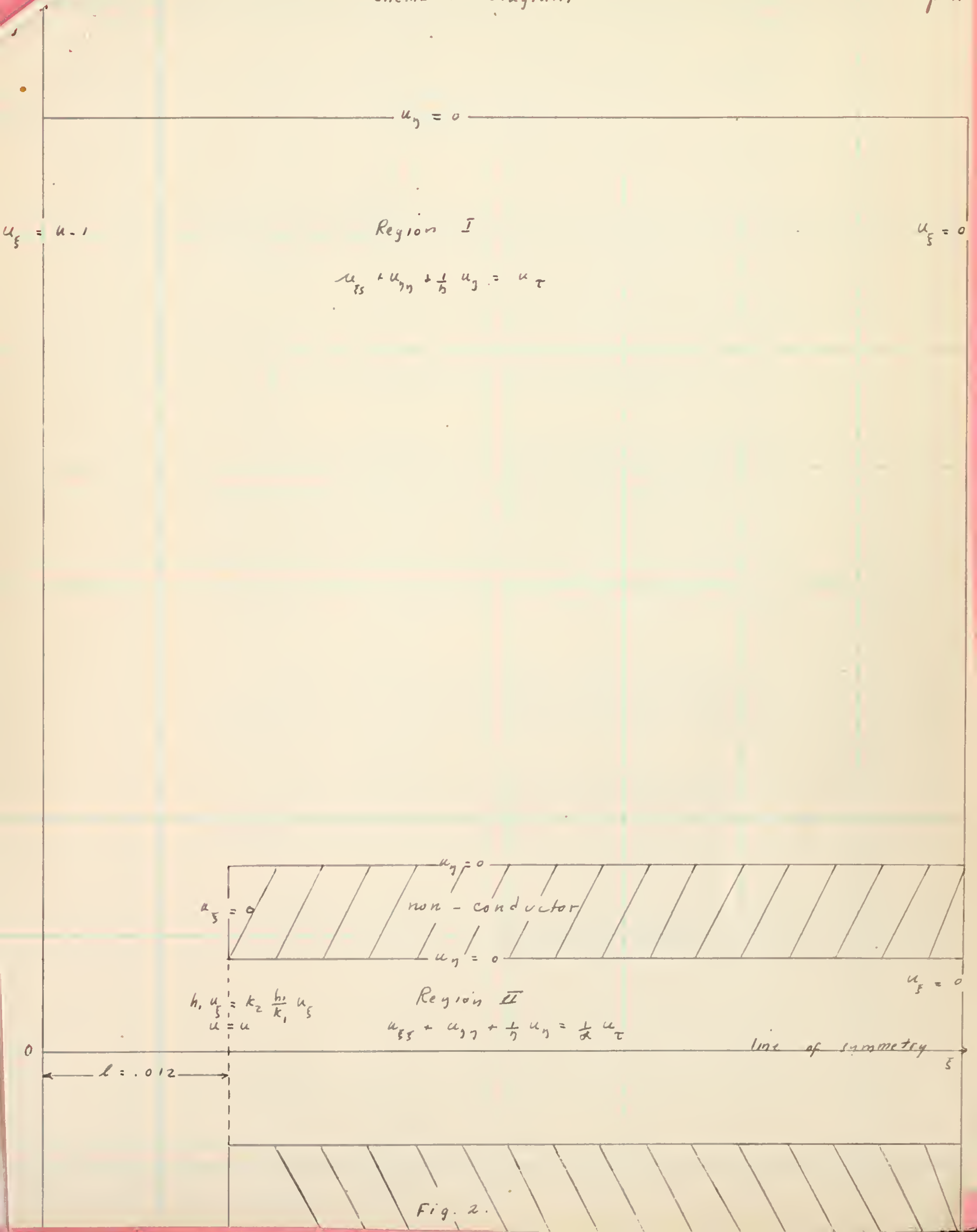
$u_\eta = 0$
$u_\xi = u - 1$
Region I
$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{\eta} u_\eta = u_\tau$
$u_\xi = 0$
$u_\eta = 0$
$u_\xi = 0$
non - conductor
$u_\eta = 0$
$u_\xi = 0$
$h_1 u_\xi = k_2 \frac{h_1}{k_1} u_\xi$
$u = u$
Region II
$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{\eta} u_\eta = \frac{1}{\alpha} u_\tau$
0
$l = .012$
line of symmetry
ξ

Fig. 2.

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